Blowout bifurcation and stability of marginal synchronization of chaos

A. Krawiecki and S. Matyjaśkiewicz
Faculty of Physics, Warsaw University of Technology, Koszykowa 75, PL-00-662 Warsaw, Poland
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Blowout bifurcations are investigated in a symmetrized extension of the replacement method of chaotic synchronization which consists of coupling chaotic systems via mutually shared variables. The coupled systems are partly linear with respect to variables that are not shared, and that form orthogonal invariant manifolds in the composite system. If the coupled systems are identical, marginal (projective) synchronization between them occurs. Breaking the symmetry by a small variation of the system parameters leads to a new kind of blowout bifurcation in which the transverse stability is exchanged between the orthogonal invariant manifolds. This bifurcation is neither supercritical nor subcritical. The latter scenarios are also observed as the parameters are further varied, leading to on-off intermittency and the appearance of riddled basins of attraction. Examples using well-known chaotic models are presented.

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I. INTRODUCTION

Systems that possess chaotic attractors contained within invariant manifolds whose dimension is less than the dimension of the system phase space can exhibit blowout bifurcation [1,2]. This bifurcation consists of the loss of stability of the above-mentioned attractor with respect to perturbations transverse to the invariant manifold at a critical value of some bifurcation parameter. The bifurcation parameter is called normal if the dynamics constrained to the invariant manifold is independent on it, and by varying this parameter only the dynamics outside the invariant manifold is modified; otherwise it is called non-normal [2]. There are two typical blowout scenarios: the supercritical (nonhysteretic) and subcritical (hysteretic) one [1]. The supercritical scenario occurs if before the blowout there is no stable attractor of the system apart from that within the invariant manifold. Then above the blowout a new attractor is formed that encompasses the attractor within the invariant manifold, and on-off intermittency occurs [3–5]: the phase trajectory stays for long times close to the invariant manifold (laminar phases) and occasionally departs from it (bursts). Before the blowout, transient on-off intermittency [6] can be observed: the phase trajectory approaches in general the invariant manifold, but initially bursts typical of on-off intermittency can be seen. The subcritical scenario occurs if before the blowout there is a stable attractor distant from that within the invariant manifold. Above the blowout only the former attractor is stable. Before the blowout both attractors are stable, but the basin of the attractor contained within the invariant manifold is riddled [7–9], i.e., in any neighborhood of a point belonging to this basin there is a positive measure set of points belonging to the basin of the other, distant attractor. It is also possible that there are two or more attractors contained within different (usually parallel) invariant manifolds of the system. Before the blowout these attractors are stable and their basins are mutually riddled (intermingled), while above the blowout they lose transverse stability and two-state (or multistate) on-off intermittency is observed [10–12]. The blowout scenario (supercritical or subcritical) that occurs in a given system is determined by the nonlinearities that control the transverse dynamics in the neighborhood of the invariant manifold [13].

In this paper connection between the blowout bifurcation and a particular kind of chaotic synchronization called marginal (or projective) synchronization (MS) [14–19] is studied (for a review of chaotic synchronization see Refs. [20,21]). The simplest form of chaotic synchronization is identical synchronization in which trajectories of properly coupled identical chaotic systems started with different initial conditions approach each other after all transients die out [22–24]. In contrast, attractors of marginally synchronized systems are shifted or scaled copies of each other; in this paper only the latter situation will be considered (sized MS). MS can be achieved using the drive-response replacement method of synchronization [22,23] if one can choose the response system as a partly linear subsystem of the drive system and force it with a chaotic variable from the drive. In the case of sized MS the drive system variables form then an invariant manifold in the phase space of the composite system, consisting of the drive and response parts. This kind of synchronization is very sensitive to mismatch between the parameters of the drive and response systems and occurs only if the corresponding parameters are equal. Otherwise, the attractor in the above-mentioned invariant manifold can lose transverse stability via the blowout bifurcation as the drive or response system parameters are even slightly varied [25]. Before the blowout the response system variables decay to zero, above the blowout they diverge to infinity.

Here a symmetrized extension of the drive-response replacement method of synchronization is considered. Coupled chaotic systems are studied that share a common variable so that each system plays simultaneously a role of the drive and response system to its counterpart. These systems are chosen to be partly linear with respect to the variables which are not shared; if the parameters of both systems are equal they show sized MS. The subspaces built of the variables of the component systems form mutually orthogonal invariant manifolds of the composite system. Blowout bifurcations from these manifolds which appear under small changes of the system parameters are investigated. A different kind of blowout bifurcation is observed that consists of the exchange of stability of the attractors in orthogonal invariant manifolds as a chosen parameter of one component system crosses the value of its counterpart in the other system; MS occurs just at the blowout. This bifurcation is neither supercritical nor subcritical, but both typical blowout scenarios are also observed as the parameters are further varied. Besides, the exchange of stability of attractors can be also caused by the loss of linear transverse stability of the invariant manifold rather than the blowout bifurcation. The examples studied numerically are based on well-known chaotic models, but the observed effects should be common in systems using the synchronization scheme proposed in this paper.

II. FORMULATION OF THE PROBLEM

Let us consider a chaotic system $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x}(t),p)$, where p is a control parameter, and the vector of state variables $\mathbf{x}$ is divided in two parts, $\mathbf{x}=(\mathbf{u},\mathbf{v})$, fulfilling the equations $\dot{\mathbf{u}}=\mathbf{f}_{\mathbf{u}}(\mathbf{u},\mathbf{v})$, $\dot{\mathbf{v}}=\mathbf{f}_{\mathbf{v}}(\mathbf{u},\mathbf{v},p)$ with $\mathbf{f}=(\mathbf{f}_{\mathbf{u}},\mathbf{f}_{\mathbf{v}})$. The symmetrized extension of the drive-response replacement method considered in this paper consists of coupling two such systems via a vector of shared variables $\mathbf{u}$,

$$\begin{aligned} \dot{\mathbf{v}}_1&=\mathbf{f}_{\mathbf{v}}(\mathbf{u},\mathbf{v}_1,p_1),\\ \dot{\mathbf{u}}&=\mathbf{f}_{\mathbf{u}}(\mathbf{u},\mathbf{v}_1,\mathbf{v}_2),\\ \dot{\mathbf{v}}_2&=\mathbf{f}_{\mathbf{v}}(\mathbf{u},\mathbf{v}_2,p_2), \end{aligned} \tag{1}$$

where the variables $\mathbf{v}_1$, $\mathbf{v}_2$ enter the equation for $\mathbf{u}$ in the same way, the functional form of the right-hand side (rhs) of the equations for $\mathbf{v}_1$, $\mathbf{v}_2$ is identical, and the parameters p_1, p_2 are independent. There are two chaotic subsystems of Eq. (1): $\mathbf{x}_1=(\mathbf{u},\mathbf{v}_1)$ and $\mathbf{x}_2=(\mathbf{u},\mathbf{v}_2)$. This is different from the situation in the drive-response replacement method in which the equation for $\mathbf{u}$ does not depend on $\mathbf{v}_2$, there is only one chaotic subsystem (the drive system) $\mathbf{x}_1=(\mathbf{u},\mathbf{v}_1)$, and the subsystem $\mathbf{v}_2$ (the response system driven by the variable $\mathbf{u}$) is not a chaotic subsystem [22,23]. In Eq. (1) the subsystems $\mathbf{x}_1$, $\mathbf{x}_2$ play simultaneously the role of both drive and response systems.

Henceforth it is assumed that in Eq. (1) equations for $\mathbf{v}_1$, $\mathbf{v}_2$ are linear with respect to these variables and invariant under scaling transformations $\mathbf{v}_{1,2}=A\mathbf{v}_{1,2}$, $A=\text{const}$. Provided that $p_1=p_2=p$, in the framework of the drive-response replacement method the variables $\mathbf{v}_1$, $\mathbf{v}_2$ would be then marginally synchronized, i.e., $\mathbf{v}_2=\alpha\mathbf{v}_1$ with the constant α dependent on initial conditions, but not on time. It turns out that the same is true for the symmetrized synchronization method (1). MS is characterized by a zero value of the conditional Lyapunov exponent [16] $\lambda_c=\lim_{t\to\infty}[|\Delta\mathbf{v}(t)|/|\Delta\mathbf{v}(0)|]$, where the vector $\Delta\mathbf{v}=\mathbf{v}_2-\mathbf{v}_1$ obeys the equation $\Delta\dot{\mathbf{v}}=\hat{D}_{\mathbf{v}}\mathbf{f}_{\mathbf{v}}|_{\mathbf{v}_1,p}\Delta\mathbf{v}=\mathbf{f}_{\mathbf{v}}(\mathbf{u},\Delta\mathbf{v},p)$; here, $\hat{D}_{\mathbf{v}}\mathbf{f}_{\mathbf{v}}$ denotes the Jacobian of $\mathbf{f}_{\mathbf{v}}$ with respect to $\mathbf{v}$, and the last equality results from the assumed linearity of $\mathbf{f}_{\mathbf{v}}$ with respect to $\mathbf{v}$.

Under the above-mentioned assumption the two chaotic subsystems $\mathbf{x}_1$, $\mathbf{x}_2$ of the system (1) become invariant subsystems with attractors contained within orthogonal invariant manifolds $\mathbf{v}_2=0$, $\mathbf{v}_1=0$, respectively. We are interested in the blowout bifurcations from these attractors. The transverse stability of, e.g., the first attractor is determined by the transverse Lyapunov exponent [1,2] $\lambda_\perp^{(1)}=\lim_{t\to\infty}[|\mathbf{w}_1(t)|/|\mathbf{w}_1(0)|]$, where $\mathbf{w}_1$ is a small deviation normal to the manifold $\mathbf{v}_2=0$. This deviation obeys the equation $\dot{\mathbf{w}}_1=\hat{D}_{\mathbf{v}}\mathbf{f}_{\mathbf{v}}|_{\mathbf{v}=0,p_2}\mathbf{w}_1=\mathbf{f}_{\mathbf{v}}(\mathbf{u},\mathbf{w}_1,p_2)$, where $\mathbf{u}$ is given by the dynamics of the subsystem $\mathbf{x}_1$ with $\mathbf{v}_2=0$. Note that identical value of $\lambda_\perp^{(1)}$ would be obtained in the drive-response replacement method if $\mathbf{x}_1$ and $\mathbf{v}_2$ were treated as the drive and response systems, respectively. The attractor within the manifold $\mathbf{v}_2=0$ is transversely stable if $\lambda_\perp^{(1)}<0$ and unstable if $\lambda_\perp^{(1)}>0$. If the parameters p_1 or p_2 are varied the exponent can change sign from negative to positive, which corresponds to the loss of transverse stability of the attractor via the blowout bifurcation. Due to the symmetry of Eq. (1) the same is true for the attractor within the manifold $\mathbf{v}_1=0$, characterized by the exponent $\lambda_\perp^{2}$.

The kind of blowout bifurcation in Eq. (1) we are interested in takes place at $p_1=p_2$, i.e., when the subsystems $\mathbf{x}_1$, $\mathbf{x}_2$ are symmetric. Then, in fact, $\lambda_\perp^{(1)}=\lambda_\perp^{(2)}=0$ since the transverse Lyapunov exponents are equal to the conditional Lyapunov exponent obtained for the drive-response replacement method. If the system symmetry is broken and, e.g., p_2 is varied from p_1 by δp then $\lambda_\perp^{(1)}\approx(d\lambda_\perp^{(1)}/dp_2)|_{p_2=p_1}\delta p$ changes sign and the transverse stability of the attractor within the manifold $\mathbf{v}_2=0$ is also changed. The bifurcation parameter p_2 is, however, non-normal, and by varying it also the transverse stability of the other attractor contained within the orthogonal invariant manifold $\mathbf{v}_1=0$ is affected. By inspection of Eq. (1), treating in turn the subsystem $\mathbf{v}_1$ as the response and $\mathbf{x}_2$ as the drive system, it is clear that the sign of $\lambda_\perp^{(2)}$ is changed at $p_1=p_2$ in the opposite way to that of $\lambda_\perp^{(1)}$. Thus by varying p_2 in the vicinity of p_1 the attractor within one of the two invariant manifolds loses stability via the blowout bifurcation and that within the other one gains stability via the inverse blowout bifurcation. The same is true if p_1 is varied in the vicinity of p_2. Hence, as a result of two simultaneous blowout bifurcations, the exchange of stability of the attractors contained within orthogonal invariant manifolds takes place. If there are no other stable attractors of Eq. (1) then before and after the blowout the phase trajectory is attracted to two different, orthogonal invariant manifolds. As usually, these manifolds are approached via transient on-off intermittency. The intermediate state just at the blowout is the one with MS between $\mathbf{x}_1$ and $\mathbf{x}_2$. This exchange-of-stability bifurcation is neither supercritical nor subcritical (since there is always only one stable attractor within one invariant manifold), and it can occur since the dynamics in the directions perpendicular to the invariant manifolds (described by $\mathbf{w}_1$, $\mathbf{w}_2$) is linear. It should be noted that blowout bifurcations that do not follow exactly any of the two typical scenarios have been also described by other authors [26–30].

Apart from the above-mentioned scenario, in the system (1) supercritical and subcritical blowout bifurcations can also appear. They take place when the values of the bifurcation

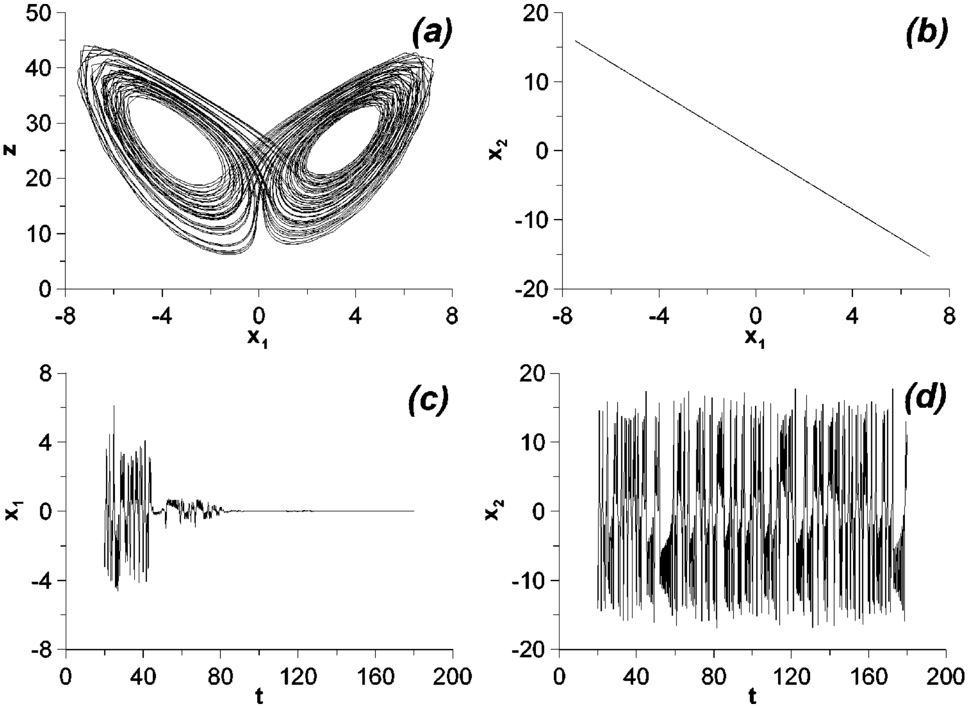

FIG. 1. (a) Chaotic attractor of the Lorenz system with $\sigma=10$, $b=8/3$, $r=28$. (b) x_1 vs x_2 for the system (2) with $r_1=r_2=28$; the linear dependence of these variables indicates sized MS. (c) Time series of x_1 for the system (2) with $r_1=28$, $r_2=28.05$; the decay of x_1 to zero shows occasional bursts typical of transient on-off intermittency; (d) Time series of x_2 for r_1, r_2 as in (c).

parameters p_1, p_2 are not equal at the blowout. The supercritical scenario occurs when after the bifurcation attractors in both orthogonal invariant manifolds are transversely unstable; a new attractor stuck to one of the manifolds is then formed and on-off intermittency in the variables $\mathbf{v}_1$ or $\mathbf{v}_2$ is observed. The subcritical scenario occurs if before the bifurcation attractors in both invariant manifolds are transversely stable; their basins of attraction are then intermingled. Besides, it was also observed that in certain cases the exchange of stability of attractors in orthogonal invariant manifolds at $p_1=p_2$ can take place via the loss and gain of the linear transverse stability of the manifolds rather than via the blowout bifurcation. In this case, if the two bifurcation parameters do not coincide, the distance from the stable manifold decreases exponentially and transient on-off intermittency is not observed.

III. NUMERICAL RESULTS

A. The Lorenz system

As a first example we consider coupling of two Lorenz systems [31]

$$\dot{x}_1=\sigma(y_1-x_1),\quad \dot{y}_1=(r_1-z)x_1-y_1,$$

$$\dot{z}=x_1y_1+x_2y_2-bz, \tag{2}$$

$$\dot{x}_2=\sigma(y_2-x_2),\quad \dot{y}_2=(r_2-z)x_2-y_2,$$

where $\sigma=10$, $b=8/3$. In the notation of Sec. II $\mathbf{u}=(z)$, $\mathbf{v}_{1,2}=(x_{1,2},y_{1,2})$, the two invariant chaotic subsystems are $\mathbf{x}_{1,2}=(x_{1,2},y_{1,2},z)$, the corresponding invariant manifolds are $\mathbf{v}_{2,1}=0$, and $r_{1,2}$ are the bifurcation parameters varied in the vicinity of $r=28$. For this set of parameters the Lorenz system is chaotic [Fig. 1(a)]. If $r_1=r_2=28$ the subsystems $\mathbf{v}_1$ and $\mathbf{v}_2$ show sized MS [Fig. 1(b)] [14–19]. If any of the two bifurcation parameters is varied, with the other kept constant, the blowout bifurcation leading to the exchange of stability of attractors contained within the orthogonal invariant manifolds takes place at $r_1=r_2$. This can be seen in Fig. 2 where the transverse Lyapunov exponent $\lambda_\perp^{(1)}$ is shown for two cases: $r_1=28$ and r_2 varied, and $r_2=28$ and r_1 varied. This exponent changes sign at $r_1=r_2$ independently of which a parameter is changed, which corresponds to the blowout bifurcation from the manifold $\mathbf{v}_2=0$. It can be seen that $\lambda_\perp^{(1)}<0$ and thus the manifold $\mathbf{v}_2=0$ is transversely stable if $r_2<r_1$, and $\lambda_\perp^{(1)}>0$ and thus the manifold $\mathbf{v}_2=0$ is transversely unstable if $r_2>r_1$. An identical plot, with the role of r_1 and r_2 inverted, can be obtained for $\lambda_\perp^{(2)}$, which yields the transverse stability condition for the manifold $\mathbf{v}_1=0$ as r_1

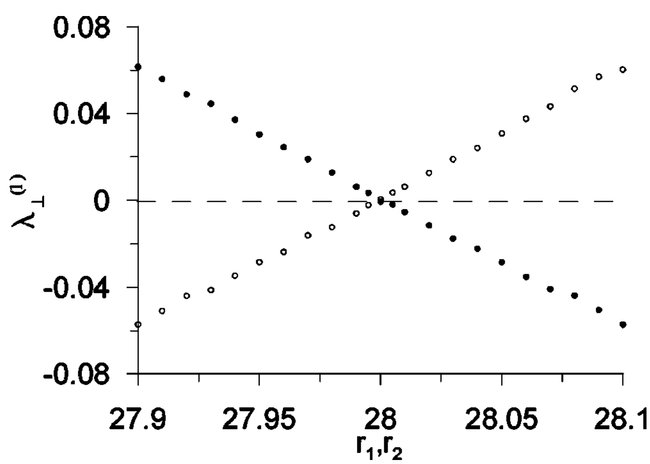

FIG. 2. Transverse Lyapunov exponent $\lambda_\perp^{(1)}$ for the system (2) vs r_1 with $r_2=28$ (dots) and vs r_2 with $r_1=28$ (circles); $\sigma=10$, $b=8/3$.

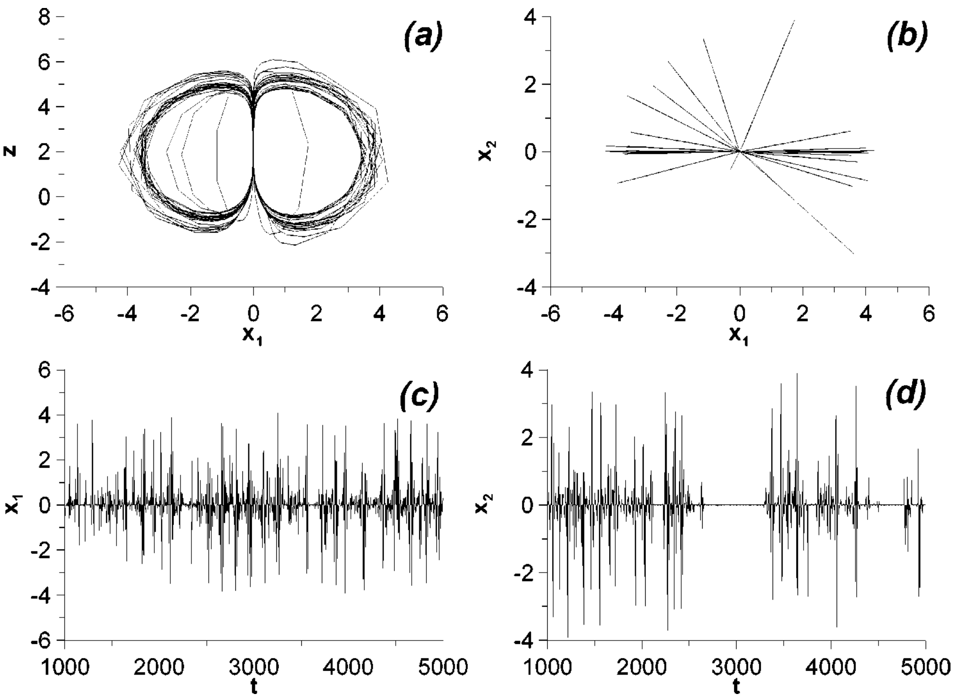

FIG. 3. (a) Chaotic attractor of the disk dynamo model with $\mu=1.7$, $\gamma=0.5$. (b) x_1 vs x_2 for the system (5) with $\gamma_1=0.5$, $\gamma_2=0.44$. (c) Time series of x_1 and (d) of x_2 for γ_1, γ_2 as in (b); the time series in (d) shows on-off intermittency.

$<r_2$. Hence at $r_1=r_2$ the stability of the two orthogonal invariant manifolds is exchanged. For $r_1\neq r_2$ the transversely stable attractors $\mathbf{x}_1\neq 0$ or $\mathbf{x}_2\neq 0$ contained within the invariant manifolds turn out to be global attractors of Eq. (2), i.e., there are no other stable attractors in a wide range of the bifurcation parameters. In particular, there are no blowout bifurcations from the two invariant subspaces other than the above-mentioned one. The variables transverse to the stable manifold exhibit transient on-off intermittency and decay to zero [Fig. 1(c)] while the variables contained within the stable manifold show behavior typical of the Lorenz system [Fig. 1(d)].

The dependence of $\lambda_\perp^{(1)}$ on r_2 can be at least qualitatively understood [25] from the analysis of time-dependent eigenvalues of the Jacobian $\hat{D}_{\mathbf{v}}\mathbf{f}_{\mathbf{v}}|_{\mathbf{v}=0,r_2}$, which are

$$\lambda_\pm(r_2,t)=\frac{1}{2}\{-(\sigma+1)\pm\sqrt{(\sigma+1)^2-4\sigma[z(t)-(r_2-1)]}\}. \tag{3}$$

Since $\mathrm{Re}\,\lambda_-\leqslant\mathrm{Re}\,\lambda_+$ the dynamics transverse to the invariant manifold $\mathbf{v}_2=0$ is mainly controlled by the real part of λ_+. Thus it can be assessed that $\lambda_\perp^{(1)}(r_2)\approx\langle\mathrm{Re}\,\lambda_+(r_2,t)\rangle$, where the angular brackets denote the time average. Assuming that r_2 is varied in the vicinity of $r_1=\mathrm{const}$ by $\delta r=r_2-r_1$ and keeping in mind that $\lambda_\perp^{(1)}(r_1)=0$ the value of $\lambda_\perp^{(1)}(r_1+\delta r)$ can be in turn approximated as

$$\begin{aligned}\lambda_\perp^{(1)}&(r_1+\delta r)\\ &\approx\left\langle \mathrm{Re}\frac{d\lambda_+}{dr_2}\bigg|_{r_1}\right\rangle\delta r\\ &=\langle\mathrm{Re}\{(\sigma+1)^2-4\sigma[z(t)-(r_2-1)]\}^{-1/2}\sigma\rangle\,\delta r.\end{aligned} \tag{4}$$

It follows from Eqs. (3) and (4) that close to the bifurcation point $r_1=r_2$ there are temporary chaotic fluctuations of the transverse Lyapunov exponent around the mean value. In particular, at the bifurcation point $\lambda_\perp^{(1)}$ is equal to zero not exactly, but only on average. Also if the exponent is on average negative, its temporary value can be positive which leads to transient on-off intermittency shown in Fig. 1(c). Such behavior of the transverse Lyapunov exponent is typical of the blowout bifurcation [1,8]. Since the time average in Eq. (4) is positive the sign of the exponent is determined by the sign of δr, hence if $r_2<r_1$ there is $\lambda_\perp^{(1)}<0$ and the invariant manifold $\mathbf{v}_2=0$ is stable, in agreement with the results of numerical simulation (Fig. 2). A similar analysis can be performed for the dependence of $\lambda_\perp^{(2)}$ on r_1. In contrast, neither the dependence of $\lambda_\perp^{(1)}$ on r_1 nor $\lambda_\perp^{(2)}$ on r_2 can be easily predicted from Eq. (3) since then the variable $z(t)$ is modified and the dependence of the transverse Lyapunov exponents on this variable is nontrivial.

B. The disk dynamo model

A second example is a set of two coupled disk dynamo models [32]

$$\dot{x}_1=zy_1-\mu_1x_1,\quad \dot{y}_1=(z-\gamma_1)x_1-\mu_1y_1,$$
$$\dot{z}=1-x_1y_1-x_2y_2, \tag{5}$$
$$\dot{x}_2=zy_2-\mu_2x_2,\quad \dot{y}_2=(z-\gamma_2)x_2-\mu_2y_2,$$

where the division of variables among chaotic subsystems and the invariant manifolds are the same as in Eq. (2), and $\gamma_{1,2}$ and $\mu_{1,2}$ are two sets of independent bifurcation parameters. For $\gamma_1=\gamma_2=0.5$, $\mu_1=\mu_2=1.7$ the two subsystems

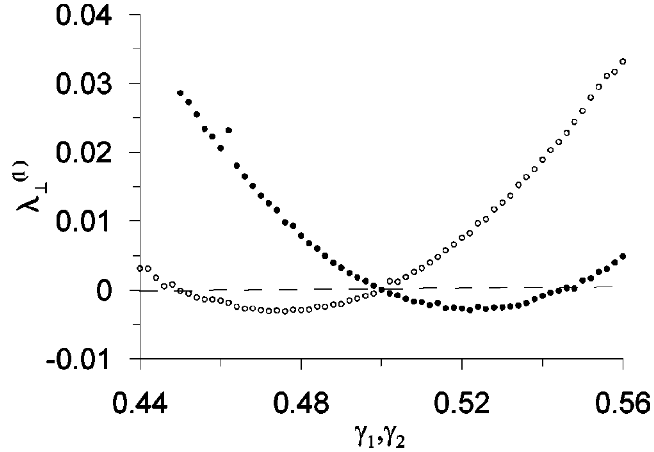

FIG. 4. Transverse Lyapunov exponent $\lambda_\perp^{(1)}$ for the system (5) vs γ_1 with $\gamma_2=0.5$ (dots) and vs γ_2 with $\gamma_1=0.5$ (circles); $\mu=1.7$.

are chaotic [the attractor is shown in Fig. 3(a)] and show sized MS [19].

Let us first fix $\mu_1=\mu_2=1.7$ and vary γ_1, γ_2 in the vicinity of 0.5. In Fig. 4 the transverse Lyapunov exponent $\lambda_\perp^{(1)}$ is shown for two cases: $\gamma_1=0.5$ and γ_2 varied, and $\gamma_2=0.5$ and γ_1 varied. For $\gamma_1=\gamma_2$ there is a blowout bifurcation leading to the exchange of stability of invariant manifolds. In the neighborhood of the bifurcation point the $\mathbf{v}_2=0$ manifold is transversely stable if $\gamma_2<\gamma_1$ and, by symmetry, the $\mathbf{v}_1=0$ manifold is transversely stable if $\gamma_2>\gamma_1$. However, in contrast with the case of the Lorenz system, the invariant manifolds can lose transverse stability also if $\gamma_1\neq\gamma_2$. For example if $\gamma_1=0.5$ and γ_2 is decreased below 0.45, $\lambda_\perp^{(1)}$ changes sign from negative to positive, which corresponds to the loss of transverse stability of the manifold $\mathbf{v}_2=0$ via the supercritical blowout bifurcation. For $\gamma_2<0.45$ both subsystems $\mathbf{x}_1$, $\mathbf{x}_2$ are excited but do not show MS [Fig. 3(b)]. Though the parameter γ_2 is non-normal, just above the blowout the chaotic dynamics on the invariant manifold is almost unchanged [Fig. 3(c)] while the dynamics transverse to the invariant manifold is typical of on-off intermittency [Fig. 3(d)]. Another supercritical blowout bifurcation from the manifold $\mathbf{v}_2=0$ can be observed for $\gamma_2=0.5$ and $\gamma_1>0.546$, and symmetric bifurcations from the manifold $\mathbf{v}_1=0$ are also possible.

If $\gamma_1=\gamma_2=0.5$ are fixed and μ_1, μ_2 are varied in the vicinity of 1.7 a different bifurcation is observed. At $\mu_2=\mu_1$ the exchange of stability of the invariant manifolds takes place: if $\mu_2>\mu_1$ the manifold $\mathbf{v}_2=0$ is transversely stable and if $\mu_2<\mu_1$ the manifold $\mathbf{v}_1=0$ is transversely stable. The respective transverse Lyapunov exponents also change signs. In this case, however, the manifolds exchange stability via the loss and gain of linear transverse stability rather than via the blowout bifurcations. This can be seen if the dynamics transverse to the stable manifold is observed for $\mu_1\neq\mu_2$: instead of transient on-off intermittency the normal variables decay exponentially to zero (Fig. 5).

The difference between the two above-mentioned exchange-of-stability bifurcations can be again qualitatively understood by looking at the eigenvalues of $\hat{D}_{\mathbf{v}}\mathbf{f}_{\mathbf{v}}|_{\mathbf{v}=0,\gamma_2,\mu_2}$

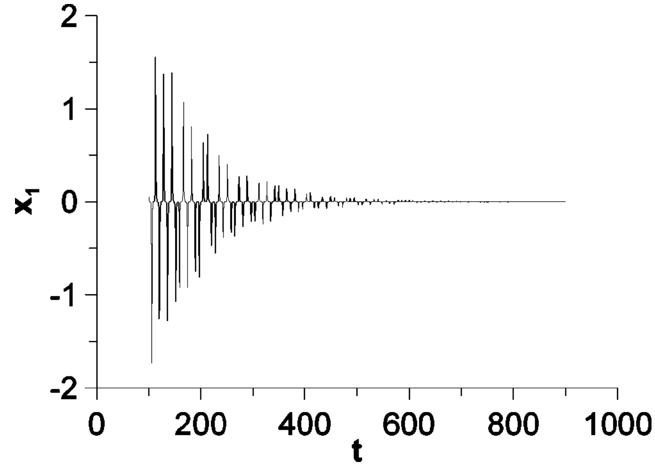

FIG. 5. Time series of x_2 for the system (5) with $\gamma_1=\gamma_2=0.5$, $\mu_1=1.7$, $\mu_2=1.69$; exponential decay of x_1 to zero can be seen.

$$\lambda_\pm(\gamma_2,\mu_2,t)=-\mu_2\pm\sqrt{z(t)[z(t)-\gamma_2]}. \tag{6}$$

Making the same approximations as in Sec. III A we arrive at

$$\begin{aligned}\lambda_\perp^{(1)}(\gamma_1+\delta\gamma,\mu_1+\delta\mu)&\\ \approx\left\langle\left.\frac{\partial\lambda_+}{\partial\gamma}\right|_{\gamma_1,\mu_1}\right\rangle\delta\gamma&+\left\langle\left.\frac{\partial\lambda_+}{\partial\mu}\right|_{\gamma_1,\mu_1}\right\rangle\delta\mu\\ =-\frac{1}{2}\langle\{z(t)[z(t)-\gamma_1]\}^{-1/2}z(t)\rangle\delta\gamma&-\delta\mu.\end{aligned} \tag{7}$$

Assuming $\delta\gamma\neq0$ the exponent becomes nonzero on average and exhibits temporary fluctuations around its mean value, as usually close to the blowout bifurcation. However, assuming $\delta\mu\neq0$ results in the exponential decay or rise of small transverse deviations $\mathbf{w}_1$ from the invariant manifold, $\mathbf{w}_1\propto\exp(-\delta\mu\ t)$ rather than the transient on-off intermittency.

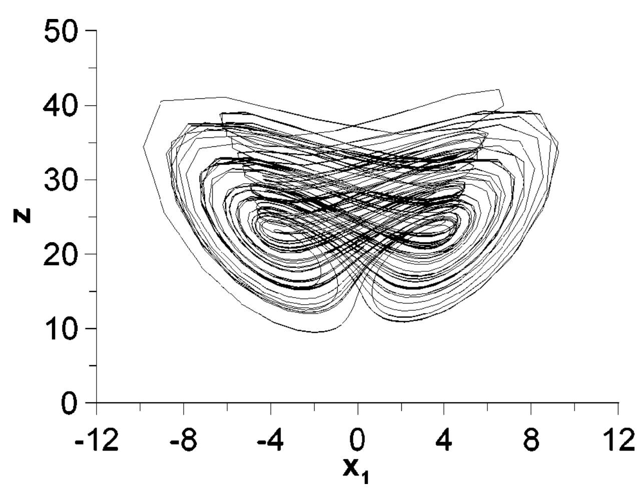

FIG. 6. Chaotic attractor of the Chen-Ueta system with $a=35$, $b=3$, $c=28$.

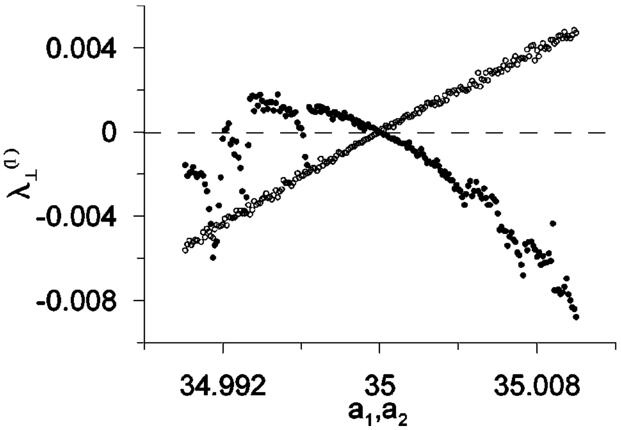

FIG. 7. Transverse Lyapunov exponent $\lambda_\perp^{(1)}$ for the system (8) vs a_1 with $a_2=35$ (dots) and vs a_2 with $a_1=35$ (circles); $b=3$, $c=28$.

C. The Chen-Ueta system

As a third example we consider coupling of two systems recently proposed by Chen and Ueta [33]

$$\dot{x}_1=a_1(y_1-x_1),\quad \dot{y}_1=(c-a_1-z)x_1+cy_1,$$
$$\dot{z}=x_1y_1+x_2y_2-bz, \tag{8}$$
$$\dot{x}_2=a_2(y_2-x_2),\quad \dot{y}_2=(c-a_2-z)x_2+cy_2,$$

where $b=3$, $c=28$, the division of variables between chaotic subsystems and the invariant manifolds are again the same as in Eqs. (2) and (5), and a_1, a_2 are bifurcation parameters varied in the vicinity of $a=35$. The attractor of Eq. (8) is shown in Fig. 6.

In Fig. 7 the transverse Lyapunov exponent $\lambda_\perp^{(1)}$ is shown for two cases: $a_1=35$ and a_2 varied, and $a_2=35$ and a_1 varied. For $a_1=a_2$ there is again a blowout bifurcation leading to the exchange of stability of invariant manifolds marked by the simultaneous change of sign of the two exponents. Apart from this bifurcation, as a_1 is varied below $a_1=34.9963$ with a_2 kept constant, a series of subcritical blowout bifurcations is observed from the manifold $\mathbf{v}_2=0$ marked by the alternating sign of $\lambda_\perp^{(1)}$. If $a_1<a_2$ then in the vicinity of the exchange-of-stability bifurcation point the manifold $\mathbf{v}_1=0$ is transversely stable and the manifold $\mathbf{v}_2=0$ is unstable. However, as a_1 is decreased, there are intervals of this parameter value for which both orthogonal invariant manifolds are transversely stable. Their basins of attraction are then intermingled (Fig. 8). The dips of the value of $\lambda_\perp^{(1)}$ below zero are not connected with the occurrence of any periodic windows in the system. It seems that the transition between the transverse stability and instability of the manifold $\mathbf{v}_2=0$ is diffuse, with the blowout bifurcation occurring over a range of the parameter a_1; a similar case was reported in Ref. [29] where it was argued that such a situation is possible if the bifurcation parameter is non-normal, as a_1 here. By symmetry, subcritical blowout bifurcation from the manifold $\mathbf{v}_1=0$ also occurs if a_1 is kept constant and a_2 is varied.

IV. DISCUSSION AND CONCLUSIONS

In this paper blowout bifurcations were investigated occurring in a symmetrized extension of the replacement method of chaotic synchronization in which coupled chaotic systems share common variables. These systems are partly linear and thus the composite system possesses attractors contained within orthogonal invariant manifolds. If the coupled systems are perfectly symmetric they are marginally synchronized. If some system parameters are varied so that the symmetry is broken, a kind of blowout bifurcation is observed in which the attractors in the orthogonal invariant subspaces exchange their transverse stability. This bifurcation is neither supercritical nor subcritical. In fact, it is a superposition of two simultaneous blowout bifurcations; in one of them an invariant manifold loses transverse stability and in the other (inverse) one another invariant manifold gains transverse stability. The transverse stability conditions for the manifolds are the same as in the drive-response synchronization method. In the latter case, however, the loss of stability of the manifold leads to the divergence of the response system variables to infinity [25] since the dynamics transverse to the invariant manifold is linear and thus unbounded. In contrast, in the case of a symmetrized synchro-

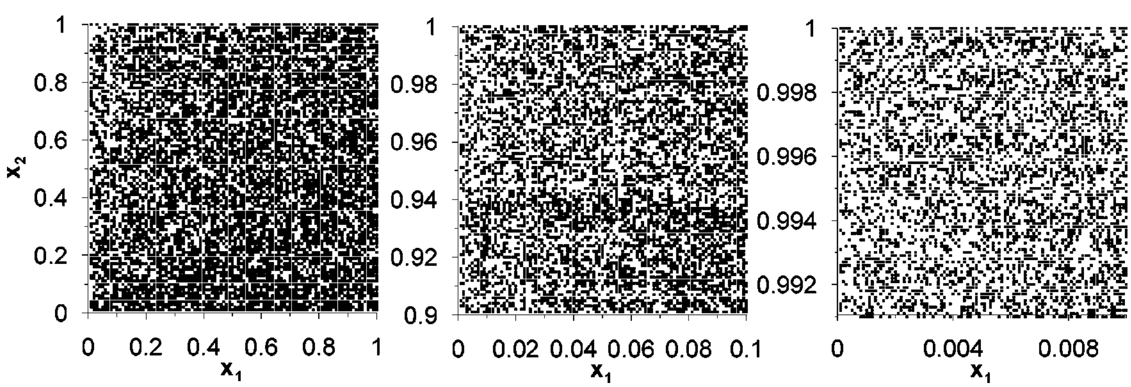

FIG. 8. From left to right: consecutive magnifications of the crossings between intermingled basins of attractors within the invariant manifolds $\mathbf{v}_2=0$ (black dots) and $\mathbf{v}_1=0$ (white dots) and the subspace $y_1=y_2=0$, $z=25$ for the system (8) with $a_1=34.993$, $a_2=35$, and other parameters as in Fig. 6; each grid contains 100×100 points.

nization scheme the phase trajectory that escapes from one manifold is captured by the orthogonal, transversely stable manifold. It is interesting to note that another example of a blowout bifurcation which is neither supercritical nor subcritical, is connected with the occurrence of cycling chaos [28].

The exchange of linear transverse stability of orthogonal invariant manifolds as well as other supercritical or subcritical blowout bifurcations leading to on-off intermittency or riddled basins were also observed in the examined systems. Thus there is a possibility of occurrence of a sequence of different blowout bifurcations as one system parameter is varied: first, e.g., the exchange-of-stability bifurcation can occur, which does not change the dimension of the attractor, followed by a supercritical bifurcation from the invariant manifold to a higher-dimensional attractor in the full phase space. A sequence of blowout bifurcations is possible since there is more than one invariant manifold in the system. Note that if more systems were connected in a way proposed in Eq. (1) there would be more invariant manifolds with different dimensions. It is interesting to check if sequences of higher-codimension blowout bifurcations from these manifolds to high-dimensional attractors can then appear [27]. This is particularly important in view of the observation of on-off intermittency [34,35] and MS [25] in high-power ferromagnetic resonance since interactions of many spin-wave pairs [modes corresponding to the component systems of Eq. (1)] can be engaged in the emergence of chaos in this case. Thus the examples studied in this paper can serve as models for the analysis of elementary blowout bifurcations which can probably appear in more complicated physical systems exhibiting marginal synchronization.

[1] E. Ott and J. Sommerer, Phys. Lett. A **188**, 39 (1994).
[2] P. Ashwin, J. Buescu, and I. Stewart, Phys. Lett. A **193**, 126 (1994).
[3] H. Fujisaka and T. Yamada, Prog. Theor. Phys. **74**, 918 (1985); **75**, 1087 (1986).
[4] N. Platt, E. Spiegel, and C. Tresser, Phys. Rev. Lett. **70**, 279 (1993).
[5] J. F. Heagy, N. Platt, and S. M. Hammel, Phys. Rev. E **49**, 1140 (1994).
[6] F. Xie and G. Hu, Phys. Rev. E **53**, 1232 (1996).
[7] J. C. Alexander, J. A. Yorke, Z. You, and I. Kan, Int. J. Bifurcation Chaos Appl. Sci. Eng. **2**, 795 (1993).
[8] E. Ott, J. C. Alexander, I. Kan, J. C. Sommerer, and J. A. Yorke, Physica D **76**, 384 (1994).
[9] Ying-Cheng Lai, C. Grebogi, J. A. Yorke, and S. C. Venkataramani, Phys. Rev. Lett. **77**, 55 (1996).
[10] Ying-Chen Lai and C. Grebogi, Phys. Rev. E **52**, R3313 (1995).
[11] J. C. Sommerer and E. Ott, Phys. Lett. A **214**, 243 (1996).
[12] Meng Zhan and Gang Hu, Phys. Rev. E **62**, 375 (2000).
[13] S. C. Venkataramani, B. R. Hunt, and E. Ott, Phys. Rev. E **54**, 1346 (1996).
[14] J. M. González-Miranda, Phys. Rev. E **53**, R5 (1996).
[15] J. Güémez and M. A. Matías, Phys. Rev. E **53**, 3059 (1996).
[16] M. A. Matías, J. Güémez, and C. Martín, Phys. Lett. A **226**, 264 (1997).
[17] J. Güémez, C. Martín, and M. A. Matías, Phys. Rev. E **55**, 124 (1997).
[18] J. M. González-Miranda, Phys. Rev. E **57**, 7321 (1998).
[19] R. Mainieri and J. Rehacek, Phys. Rev. Lett. **82**, 3042 (1999).
[20] T. Kapitaniak, *Controlling Chaos. Theoretical and Practical Methods in Nonlinear Dynamics* (Academic Press, London, 1996).
[21] L. M. Pecora, T. L. Carroll, G. A. Johnson, D. J. Mar, and J. F. Heagy, Chaos **7**, 520 (1997).
[22] L. M. Pecora and T. L. Carroll, Phys. Rev. Lett. **64**, 821 (1990).
[23] L. M. Pecora and T. L. Carroll, Phys. Rev. A **44**, 2374 (1991).
[24] J. F. Heagy, T. L. Carroll, and L. M. Pecora, Phys. Rev. E **50**, 1874 (1994).
[25] A. Krawiecki and A. Sukiennicki, Chaos, Solitons Fractals **11**, 1445 (2000).
[26] T. Kapitaniak, Yu. Maistrenko, A. Stefanski, and J. Brindley, Phys. Rev. E **57**, R6253 (1998).
[27] P. Ashwin and P. J. Aston, Phys. Lett. A **244**, 262 (1998).
[28] P. Ashwin and A. M. Rucklidge, Physica D **122**, 134 (1998).
[29] E. Covas, P. Ashwin, and R. Tavakol, Phys. Rev. E **56**, 6451 (1997).
[30] O. de Feo and R. Ferriere, Int. J. Bifurcation Chaos Appl. Sci. Eng. **10**, 443 (2000).
[31] E. N. Lorenz, J. Atmos. Sci. **20**, 130 (1963).
[32] T. Rikitake, Proc. Cambridge Philos. Soc. **54**, 89 (1958).
[33] G. Chen and T. Ueta, Int. J. Bifurcation Chaos Appl. Sci. Eng. **9**, 1465 (1999).
[34] A. Krawiecki and A. Sukiennicki, Acta Phys. Pol. A **88**, 269 (1995).
[35] F. Rödelsperger, A. Čenys, and H. Benner, Phys. Rev. Lett. **75**, 2594 (1995).